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On Young-type inequalities

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1. Introduction

The present paper is inspired by the following result of LOSONCZI [10].

Let $f, g\colon]0, \infty[\to\mathbf{R}$ be arbitrary functions. The Young-type inequality

$$xy \leqq f(x)+g(y), \quad x, y > 0 \tag{1}$$

is satisfied if and only if there exist nonnegative functions $p, q\colon]0, \infty[\to[0, \infty[$, a constant $\alpha\in\mathbf{R}$ and a Young function φ such that

$$f(x) = \int_0^x \varphi(t)\,dt+p(x)+\alpha, \quad x > 0,$$

$$g(y) = \int_0^y \varphi^{(-1)}(s)\,ds+q(y)-\alpha, \quad y > 0,$$

where $\varphi^{(-1)}$ denotes the right inverse of φ.

Here $\varphi\colon [0, \infty[\to[0, \infty[$ is called a *Young function* if it is increasing and right continuous and $\lim_{x\to\infty} \varphi(x)=\infty$. The *right inverse* of φ is defined by

$$\varphi^{(-1)}(y) = \begin{cases} 0, & \text{if } \ 0 \leqq y < \varphi(0) \\ \sup\{x \geqq 0 | \varphi(x) \leqq y\}, & \text{if } \ \varphi(0) \leqq y, \end{cases}$$

and it turns out that $\varphi^{(-1)}$ is also a Young function.

Taking $\alpha=0, p\equiv q\equiv 0$, the "if" part of the above statement reduces to

$$xy \leqq \int_0^x \varphi(t)\,dt + \int_0^y \varphi^{(-1)}(s)\,ds, \quad x, y > 0 \tag{2}$$

which is called Young's inequality although YOUNG [15] proved it only when the

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derivatives of φ and $\varphi^{(-1)}$ exist everywhere. There are several generalizations of this inequality. Here we mention only papers of Birnbaum and Orlicz [1], Boas and Marcus [2], [3], [4], Cooper [5], Cunningham and Grossman [6], Dankert and König [7], Diaz and Metcalf [8], Klambauer [9], Oppenheim [12] and Zaanen [15].

The "only if" part of the above result of Losonczi states that (1) can always be obtained by weakening a Young's inequality, in other words, this means that the Young inequalities are the only essential inequalities of the form (1).

In what follows, we deal with the functional inequality

$$H(x,y) \leqq f(x)+g(y), \quad a \leqq x \leqq A, \quad b \leqq y \leqq B,$$

where H is a given function and f, g are unknown functions. For a large class of functions H we prove an analogue of the theorem of Losonczi. The only point where our results are not more general than that of Losonczi [10] is that we assume x and y to be in the closed intervals $[a, A]$ and $[b, B]$, respectively.

2. Young functions

Let $[a, A]$ and $[b, B]$ be given fixed intervals throughout this paper. A function $\varphi\colon [a, A]\to[b, B]$ is called a *Young function* (cf. Losonczi [10], [11], Cunningham and Grossman [6]) if

(i) φ is increasing and right continuous,

(ii) $\varphi(A)=B$.

The *right inverse* of φ, is the function $\varphi^{(-1)}\colon [b, B]\to[a, A]$ defined by

$$\varphi^{(-1)}(y) = \begin{cases} a, & \text{if } \ b \leqq y < \varphi(a) \\ \sup\{x \geqq a \mid \varphi(x) \leqq y\}, & \text{if } \ y \geqq \varphi(a). \end{cases}$$

It is easy to see that $\varphi^{(-1)}$ is also a Young function (i.e., it is increasing, right continuous and $\varphi^{(-1)}(B)=A$), furthermore, the right inverse of $\varphi^{(-1)}$ equals φ.

A Young function $\varphi\colon [a, A]\to[b, B]$ is called *elementary* if there exist $a=:x_0< <x_1<\ldots<x_{n-1}<x_n:=A$ and $b\leqq y_1<\ldots<y_n\leqq B$ such that

$$\varphi(x) = y_i \quad \text{if} \quad x_{i-1} \leqq x < x_i \quad (i = 1, \ldots, n).$$

Then the right inverse of φ is also an elementary Young function and

$$\varphi^{(-1)}(y) = \begin{cases} a, & \text{if } \ b \leqq y < y_1, \\ x_i, & \text{if } \ y_i \leqq y < y_{i+1} \quad (i = 1, \ldots, n-1), \\ A, & \text{if } \ y_n \leqq y \leqq B. \end{cases}$$

We shall need the following

Lemma. *Let* $\varphi\colon [a, A]\to[b, B]$ *be an arbitrary Young function. Then there exists an increasing sequence of elementary Young functions* $\varphi_n\colon [a, A]\to[b, B]$, $(n\in\mathbf{N})$ *satisfying*

(3) $$\lim_{n\to\infty}\varphi_n=\varphi \quad\text{and}\quad \lim_{n\to\infty}\varphi_n^{(-1)}=\varphi^{(-1)}.$$

Proof. Write $\alpha:=A-a$ and let $\{\tau_1, \tau_2, \dots\}$ be the set of all points in $[a, A]$ where φ is not continuous. Denote by X_n $(n\in\mathbf{N})$ the set

$$\left\{\tau_1, \dots, \tau_n, a, a+\frac{\alpha}{2^n}, \dots, a+\frac{\alpha(2^n-1)}{2^n}, A\right\}.$$

Assume that the elements of X_n are $a=x_0<\dots<x_m=A$. Then define $\varphi_n\colon [a, A]\to\to[b, B]$ in the following way

$$\varphi_n(x)=\begin{cases}\varphi(x_i), & \text{if}\quad x_i\leqq x<x_{i+1}, \quad i=0, \dots, m-1,\\ B, & \text{if}\quad x=A.\end{cases}$$

Since $X_n\subseteqq X_{n+1}$ $(n\in\mathbf{N})$, it is obvious that (φ_n) is an increasing sequence of elementary Young functions.

To prove the first equality in (3), let $a\leqq x\leqq A$ be arbitrary. If either $x=a$ or $x=A$ then $\varphi_n(x)=\varphi(x)$ therefore there is nothing to show, so we assume that $a<x<A$. If φ is not continuous at x then there exists a k such that $x=\tau_k$, i.e., $x\in X_k\cap X_{k+1}\cap\dots$. Thus $\varphi_n(x)=\varphi(x)$ if $n\geqq k$. Therefore the first equality in (3) is obvious again.

Now suppose that $a<x<A$ and that φ is continuous at x. Let $\varepsilon>0$ be arbitrary. Then there exists a $\delta>0$ such that $|x-x'|<\delta$ implies $|\varphi(x)-\varphi(x')|<\varepsilon$. Therefore, if $\alpha/2^n<\delta$, there exist two consecutive elements x' and x'' of X_n such that $x-\delta<x'\leqq x<x''$. Then $\varphi_n(x)=\varphi(x')$ and $|\varphi(x')-\varphi(x)|<\varepsilon$. Thus we have proved that $|\varphi_n(x)-\varphi(x)|<\varepsilon$ if $\alpha/2^n<\delta$, i.e., the first equality in (3) holds true in this case, too.

To prove the second equality, let $b\leqq y\leqq B$ be fixed. The inequality $\varphi_n\leqq\varphi$ yields $\varphi^{(-1)}\leqq\varphi_n^{(-1)}$. Therefore, if $\varphi^{(-1)}(y)=A$ then $\varphi_n^{(-1)}(y)=A$. Thus we may assume that $x:=\varphi^{(-1)}(y)<A$. Let $n\in\mathbf{N}$ be fixed. Then there exist two consecutive elements x' and x'' of X_n such that $x'\leqq x<x''$. If $\varphi(x'')$ were equal to $\varphi(x)$, then $\varphi^{(-1)}(y)$ would be greater than or equal to x''. Thus necessarily $\varphi(x')\leqq\varphi(x)==y<\varphi(x'')$. This yields $\varphi_n(x)=\varphi(x')<\varphi(x'')=\varphi_n(x'')$. Now, by definition, $\varphi_n^{(-1)}(y)<x''$. Since $x''-x\leqq x''-x'<\alpha/2^n$, therefore

$$\varphi_n^{(-1)}(y)<x+\alpha/2^n=\varphi^{(-1)}(y)+\alpha/2^n.$$

On the other hand we have $\varphi^{(-1)}\leqq\varphi_n^{(-1)}$, thus $|\varphi_n^{(-1)}(y)-\varphi^{(-1)}(y)|<\alpha/2^n$ holds for all $n\in\mathbf{N}$. This relation shows that $\varphi_n^{(-1)}(y)$ converges to $\varphi^{(-1)}(y)$ if $\varphi^{(-1)}(y)\neq A$.

3. Generalizations of Young's inequality

Denote by $\mathscr{H}$ the set of functions H: $[a, A]\times[b, B]$ that satisfy the inequality

$$H(x, v)+H(u, y) \leqq H(x, y)+H(u, v) \tag{4}$$

for all $a\leqq x\leqq u\leqq A$, $b\leqq y\leqq v\leqq B$. We note that if H is a C^2 function, then (4) holds if and only if

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}H(x, y) \geqq 0 \tag{5}$$

is valid for all $a\leqq x\leqq A$, $b\leqq y\leqq B$.

The following theorem gives a Young-type inequality for elementary Young functions.

Theorem 1. *Let $H\in\mathscr{H}$ and assume that H is absolutely continuous on the boundary of $[a, A]\times[b, B]$. Then H is absolutely continuous in both variables, furthermore*

$$H(x, y) \leqq H(a, b)+\int_a^x \partial_1 H(t, \varphi(t))\, dt+\int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds \tag{6}$$

holds for all elementary Young functions φ: $[a, A]\to[b, B]$ and for all $a\leqq x\leqq A$, $b\leqq y\leqq B$.

(Here $\partial_1 H$ and $\partial_2 H$ denote the partial derivatives of H with respect to the first or second variable respectively.)

Proof. Let $b\leqq y\leqq B$ be fixed. We show that $x\mapsto H(x,y)$, $a\leqq x\leqq A$ is an absolutely continuous function. Since H satisfies (4), we have

$$H(u, b)-H(x, b) \leqq H(u, y)-H(x, y) \leqq H(u, B)-H(x, B)$$

for $a\leqq x\leqq u\leqq A$. Thus we obtain

$$|H(u, y)-H(x, y)| \leqq \max\{|H(u, b)-H(x, b)|,\ |H(u, B)-H(x, B)|\} \tag{7}$$

for all $x, u\in[a, A]$. By assumption,

$$x \mapsto H(x, b) \quad\text{and}\quad x \mapsto H(x, B)$$

are absolutely continuous on $[a, A]$. Therefore, by the estimate (7), the function $x\mapsto H(x,y)$ is also absolutely continuous. Thus the partial derivative $\partial_1 H(x,y)$ exists for almost all $a\leqq x\leqq A$ (if y is fixed). (See B. Sz.-Nagy [14] for the properties of absolutely continuous functions.) A similar argument shows that $y\mapsto H(x,y)$ is also an absolutely continuous function on $[b, B]$ for each fixed $a\leqq x\leqq A$.

Let $\varphi\colon [a, A]\to[b, B]$ be an arbitrary elementary Young function. Then there exist $a=x_0<x_1<\ldots<x_n=A$ and $b\leqq y_1<\ldots<y_n\leqq B$ such that

$$\varphi(t) = y_i \quad \text{if} \quad x_{i-1} \leqq t < x_i.$$

Assume that $x_{k-1}\leqq x\leqq x_k$. Then

$$\int_a^x \partial_1 H(t, \varphi(t))\, dt = \sum_{i=1}^{k-1} \int_{x_{i-1}}^{x_i} \partial_1 H(t, \varphi(t))\, dt + \int_{x_{k-1}}^{x} \partial_1 H(t, \varphi(t))\, dt =$$

$$= \sum_{i=1}^{k-1} \int_{x_{i-1}}^{x_i} \partial_1 H(t, y_i)\, dt + \int_{x_{k-1}}^{x} \partial_1 H(t, y_k)\, dt =$$

$$= \sum_{i=1}^{k-1} \big(H(x_i, y_i) - H(x_{i-1}, y_i)\big) + H(x, y_k) - H(x_{k-1}, y_k).$$

For the sake of simplicity, we write $y_0:=b$ and $y_{n+1}:=B$. Then

$$\varphi^{(-1)}(s) = x_j \quad \text{if} \quad y_j \leqq s < y_{j+1} \quad (j = 0, \ldots, n).$$

Assume that $y_m\leqq y\leqq y_{m+1}$. Then

$$\int_0^y \partial_2 H\big(\varphi^{(-1)}(s), s\big)\, ds = \sum_{j=0}^{m-1} \int_{y_j}^{y_{j+1}} \partial_2 H\big(\varphi^{(-1)}(s), s\big)\, ds + \int_{y_m}^{y} \partial_2 H\big(\varphi^{(-1)}(s), s\big)\, ds =$$

$$= \sum_{j=0}^{m-1} \int_{y_j}^{y_{j+1}} \partial_2 H(x_j, s)\, ds + \int_{y_m}^{y} \partial_2 H(x_m, s)\, ds =$$

$$= \sum_{j=0}^{m-1} \big(H(x_j, y_{j+1}) - H(x_j, y_j)\big) + H(x_m, y) - H(x_m, y_m).$$

To prove (6), we distinguish two cases.

If $k\leqq m$, then

$$\Delta := \int_a^x \partial_1 H(t, \varphi(t))\, dt + \int_b^y \partial_2 H\big(\varphi^{(-1)}(s), s\big)\, ds + H(a, b) - H(x, y) =$$

$$= \big(H(x_k, y) + H(x, y_k) - H(x, y) - H(x_k, y_k)\big) +$$

$$+ \sum_{i=k}^{m-1} \big(H(x_{i+1}, y) + H(x_i, y_{i+1}) - H(x_i, y) - H(x_{i+1}, y_{i+1})\big).$$

Now applying (4), one can check that all the terms on the right-hand side of this equation are nonnegative. Thus (6) is valid in this case.

If $m<k$, then

$$\Delta = \big(H(x, y_k)+H(x_{k-1}, y)-H(x_{k-1}, y_k)-H(x, y)\big)+$$
$$+\sum_{i=m}^{k-2}\big(h(x_{i+1}, y_{i+1})+H(x_i, y)-H(x_i, y_{i+1})-H(x_{i+1}, y)\big)$$

and a similar argument shows that $\Delta\geqq 0$ is also satisfied. Thus (6) is proved in both cases.

Remark. One may ask whether (6) is true for all Young functions $\varphi\colon [a, A]\to[b, B]$ under the regularity assumptions of the theorem. The following example shows that it is not so: Let $H(x, y)=\min(x, y)$ and $\varphi(x)=x$ for all $x, y\in[0, 1]$. Then $H\in\mathscr{H}$ and H is absolutely continuous on the boundary of $[0, 1]\times[0, 1]$. However, the values $\partial_1 H(t, \varphi(t))$ and $\partial_2 H(\varphi^{(-1)}(s), s)$ are not defined for any $t, s\in[0, 1]$. Thus the right-hand side of (6) has no meaning. Therefore in order to prove (6) for arbitrary Young functions φ, we need stronger regularity properties of H.

Theorem 2. *Let $H\in\mathscr{H}$ and assume that the partial derivatives $\partial_1 H(x, y)$ and $\partial_2 H(x, y)$ exist for all $a\leqq x\leqq A$, $b\leqq y\leqq B$, furthermore*

$$y\mapsto\partial_1 H(x, y),\quad (b\leqq y\leqq B)\quad\text{and}\quad x\mapsto\partial_2 H(x, y),\quad (a\leqq x\leqq A)$$

are continuous functions for almost all fixed $a\leqq x\leqq A$ and $b\leqq y\leqq B$, respectively. Then (6) is satisfied for all Young functions $\varphi\colon [a, A]\to[b, B]$ and $a\leqq x\leqq A$, $b\leqq y\leqq B$.

Proof. Since H satisfies (4), we have

$$\frac{H(u, y)-H(x, y)}{u-x}\leqq\frac{H(u, v)-H(x, v)}{u-x}$$

for $a\leqq x<u\leqq A$, $b\leqq y\leqq v\leqq B$. Taking the limit $u\to x$, we get

$$\partial_1 H(x, y)\leqq\partial_1 H(x, v).$$

Therefore the function $y\mapsto\partial_1 H(x, y)$, $b\leqq y\leqq B$ is not only continuous, but it is increasing for almost all $a\leqq x\leqq A$. Similarly, $x\mapsto\partial_2 H(x, y)$ is also increasing for $a\leqq x\leqq A$.

To prove (6), let φ be an arbitrary Young function. Then, by the Lemma, there exists an increasing sequence φ_n of elementary Young functions such that (3) holds. Thus, by the above properties of H,

$$\lim_{n\to\infty}\partial_1 H(t, \varphi_n(t))=\partial_1 H(t, \varphi(t)) \tag{7}$$

for almost all $a \leqq t \leqq A$ and

$$\lim_{n\to\infty} \partial_2 H(\varphi_n^{(-1)}(s), s) = \partial_2 H(\varphi^{(-1)}(s), s)$$

for almost all $b \leqq s \leqq B$. Since

$$\partial_1 H(t, b) \leqq \partial_1 H(t, \varphi_n(t)) \leqq \partial_1 H(t, \varphi(t)) \leqq \partial_1 H(t, B),$$

furthermore $\partial_1 H(t, b)$ and $\partial_1 H(t, B)$ are integrable functions on $[a, A]$, therefore the Lebesgue convergence theorem (see B. Sz.-Nagy [14]) can be applied. Thus, by (7), we get

$$\lim_{n\to\infty} \int_a^x \partial_1 H(t, \varphi_n(t))\, dt = \int_a^x \partial_1 H(t, \varphi(t))\, dt \tag{8}$$

for all $a \leqq x \leqq A$. Similarly,

$$\lim_{n\to\infty} \int_b^y \partial_2 H(\varphi_n^{(-1)}(s), s)\, ds = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds. \tag{9}$$

However, Theorem 1 yields

$$H(x, y) \leqq H(a, b) + \int_a^x \partial_1 H(t, \varphi_n(t))\, dt + \int_b^x \partial_2 H(\varphi_n^{(-1)}(s), s)\, ds$$

for all $n \in \mathbf{N}$, $a \leqq x \leqq A$, $b \leqq y \leqq B$. Letting $n \to \infty$ and using (8) and (9) we obtain (6), which was to be proved.

Remarks.

(i) Assuming only the existence of the partial derivatives $\partial_1 H(x, y)$, $\partial_2 H(x, y)$ of $H \in \mathscr{H}$, and using the same method one can prove that

$$H(x, y) \leqq H(a, b) + \int_a^x \partial_1 H(t, \varphi(t) - 0)\, dt + \int_b^x \partial_2 H(\varphi^{(-1)}(s) + 0, s)\, ds$$

holds for all Young functions φ.

(ii) The inequality (6) can be interpretated in the following way: The equation

$$m([x, u] \times [y, v]) = H(x, y) + H(u, v) - H(x, v) - H(u, y),$$
$$a \leqq x \leqq u \leqq A, \quad b \leqq y \leqq v \leqq B$$

defines a Lebesgue—Stieltjes measure on $[a, A] \times [b, B]$. If φ is a Young function then let

$$P_{x,y} := [a, x] \times [b, y],$$
$$Q_x := \{(s, t) | a \leqq s \leqq x,\ b \leqq t \leqq \varphi(s)\},$$
$$R_y := \{(s, t) | b \leqq t \leqq y,\ a \leqq s \leqq \varphi^{(-1)}(t)\}.$$

Now one can see that $P_{x,y} \subseteq Q_x \cup R_y$ for all $a \leqq x \leqq A$, $b \leqq y \leqq B$. Therefore

$$m(P_{x,y}) \leqq m(Q_x) + m(R_y). \tag{10}$$

Calculating the measures of these sets, one can show that (10) reduces to (6). Using the above argument, (6) was proved by the author [13] in the case when H is a C^2 function, i.e., when the measure m has the density function $\partial_1 \partial_2 H(x, y)$.

4. Young-type functional inequalities

Theorem 3. *Let H satisfy the conditions of Theorem 2; furthermore, let $f\colon [a, A] \to \mathbf{R}$ and $g\colon [b, B] \to \mathbf{R}$ be arbitrary functions. Then the functional inequality*

$$H(x, y) \leqq f(x) + g(y), \quad a \leqq x \leqq A, \quad b \leqq x \leqq B \tag{11}$$

is satisfied if and only if there exist two nonnegative functions $p\colon [a, A] \to [0, \infty[$, $q\colon [b, B] \to [0, \infty[$, a constant $\alpha \in \mathbf{R}$ and a Young function $\varphi\colon [a, A] \to [b, B]$ such that

$$f(x) = \int_a^x \partial_1 H(t, \varphi(t))\, dt + p(x) + \alpha, \quad a \leqq x \leqq A, \tag{12}$$

$$g(y) = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds + q(y) + H(a, b) - \alpha, \quad b \leqq y \leqq B. \tag{13}$$

Proof. The "if" part of the statement is a consequence of Theorem 2.

To prove the converse, assume that (11) is satisfied. Define $f_1\colon [a, A] \to \mathbf{R}$ by

$$f_1(x) := \sup_y \left(H(x, y) - g(y)\right). \tag{14}$$

Then (11) yields $f_1 \leqq f$. Therefore the function $p := f - f_1$ is nonnegative. Using the subadditivity of the sup operation and the estimate (7), we get

$$f_1(x) = \sup_y \left(H(x, y) - H(u, y) + H(u, y) - g(y)\right) \leqq \sup_y \left(H(x, y) - H(u, y)\right) + f_1(u) \leqq$$
$$\leqq \max\{|H(x, b) - H(u, b)|,\ |H(x, B) - H(u, B)|\} + f_1(u),$$

whence we obtain

$$|f_1(x) - f_1(u)| \leqq \max\{|H(x, b) - H(u, b)|,\ |H(x, B) - H(u, B)\} \tag{15}$$

for all $x, u \in [a, A]$. Since H is absolutely continuous on the boundary of $[a, A] \times \times [b, B]$ therefore (15) shows that f_1 is an absolutely continuous function. By (14) we have

$$H(x, y) \leqq f_1(x) + g(y) \quad a \leqq x \leqq A, \quad b \leqq x \leqq B.$$

Therefore the function $g_1\colon [b, B] \to \mathbf{R}$, defined by

$$g_1(y) := \sup_x \big(H(x, y) - f_1(x)\big), \tag{16}$$

satisfies $g_1 \leqq g$. Thus the function $q := g - g_1$ is nonnegative. A similar argument shows that g_1 is also an absolutely continuous function, and by (16) we have

$$H(x, y) \leqq f_1(x) + g_1(y), \quad a \leqq x \leqq A, \quad b \leqq y \leqq B. \tag{17}$$

Thus

$$f_1(x) = \sup_y \big(H(x, y) - g_1(y)\big) \geqq \sup_y \big(H(x, y) - g(y)\big) = f_1(x),$$

i.e.,

$$f_1(x) = \sup_y \big(H(x, y) - g_1(y)\big) \tag{18}$$

for all $a \leqq x \leqq A$. Write

$$\Phi := \{(x, y) | H(x, y) = f_1(x) + g_1(y)\}.$$

Since $x \mapsto H(x, y) - f_1(x)$ and $y \mapsto H(x, y) - g_1(y)$ are continuous functions, therefore the supremum in (16) and (18) is attained, i.e., for all x there exists y such that $(x, y) \in \Phi$, and for all y there exists x such that $(x, y) \in \Phi$.

The following estimate shows that H is a continuous function:

$$\begin{aligned}
|H(x, y) - H(u, v)| &\leqq |H(x, y) - H(u, y)| + |H(u, y) - H(u, v)| \leqq \\
&\leqq \max\{|H(x, b) - H(u, b)|,\ |H(x, B) - H(u, B)|\} + \\
&\quad + \max\{|H(a, y) - H(a, v)|,\ |H(A, y) - H(A, v)|\}.
\end{aligned}$$

Thus Φ is a closed set. Define $\varphi\colon [a, A] \to [b, B]$ by

$$\varphi(x) = \sup\{y | (x, y) \in \Phi\}.$$

Clearly, $(x, \varphi(x)) \in \Phi$, i.e.,

$$H\big(x, \varphi(x)\big) = f_1(x) + g_1\big(\varphi(x)\big), \quad a \leqq x \leqq A. \tag{19}$$

First we show that φ is a Young function. If φ were not increasing, then there would exist x, z such that $a \leqq x < z \leqq A$ and $\varphi(x) > \varphi(z)$. Then

$$\begin{aligned}
-H\big(x, \varphi(x)\big) &= -f_1(x) - g_1\big(\varphi(x)\big), \\
H\big(x, \varphi(z)\big) &\leqq f_1(x) + g_1\big(\varphi(x)\big), \\
-H\big(z, \varphi(z)\big) &= -f_1(z) + g_1\big(\varphi(z)\big), \\
H\big(z, \varphi(x)\big) &< f_1(z) + g_1\big(\varphi(x)\big).
\end{aligned}$$

Adding these inequalities, we get

$$H\big(x, \varphi(z)\big) + H\big(z, \varphi(x)\big) - H\big(x, \varphi(x)\big) - H\big(z, \varphi(z)\big) < 0,$$

which contradicts (4). Thus φ is an increasing function.

To prove the right continuity of φ, let x_0 be arbitrary and let x_n be a decreasing sequence with $\lim_{n\to\infty} x_n = x_0$. Then $\varphi(x_n)$ is convergent, write $y_0 := \lim_{n\to\infty} \varphi(x_n)$. Φ is closed and $(x_n, \varphi(x_n)) \in \Phi$ for all $n \in \mathrm{N}$, therefore $(x_0, y_0) \in \Phi$. Thus $\varphi(x_0) \geqq y_0$. On the other hand, $\varphi(x_0) \leqq \varphi(x_n)$ for all $n \in \mathrm{N}$, whence we get $\varphi(x_0) \leqq y_0$. So $\varphi(x_0) = y_0 = \lim_{n\to\infty} \varphi(x_n)$, which was to be proved.

Finally we show that $(a, b), (A, B) \in \Phi$. If $(a, b) \notin \Phi$, then

$$H(a, b) < f_1(a) + g_1(b).$$

However, by the properties of Φ, there exist $a \leqq x \leqq A$ and $b \leqq y \leqq B$, such that

$$-H(x, b) = -f_1(x) - g_1(b),$$
$$-H(a, y) = -f_1(a) - g_1(y),$$

and we also have (17). Adding these four inequalities, we obtain

$$H(a, b) + H(x, y) - H(x, b) - H(a, y) < 0,$$

which is a contradiction. Thus $(a, b) \in \Phi$. Similarly, one sees that $(A, B) \in \Phi$. This latter relation means that $\varphi(A) = B$. Thus we have proved that φ is a Young function.

Our next aim is to verify

$$H(\varphi^{-1}(y), y) = f_1(\varphi^{-1}(y)) + g_1(y), \quad b \leqq y \leqq B. \tag{20}$$

Assume the contrary, that for a value y

$$H(\varphi^{(-1)}(y), y) < f_1(\varphi^{(-1)}(y)) + g_1(y). \tag{21}$$

Write $x := \varphi^{-1}(y)$. Then, by (19), we have

$$-H(x, \varphi(x)) = -f_1(x) - g_1(\varphi(x)). \tag{22}$$

Now we distinguish two cases. If $x = \varphi^{(-1)}(y) = a$, then $y \leqq \varphi(a)$. By the properties of Φ, there exists a value $u > x = a$ such that

$$-H(u, y) = -f_1(u) - g_1(y) \tag{23}$$

and we also have

$$H(u, \varphi(a)) \leqq f_1(u) + g_1(\varphi(a)). \tag{24}$$

Adding (21), (22), (23) and (24) we get

$$H(a, y) + H(u, \varphi(a)) - H(a, \varphi(a)) - H(u, y) < 0,$$

which is a contradiction. Therefore (20) is valid if $\varphi^{(-1)}(y) = a$.

If $x = \varphi^{(-1)}(y) > a$, then, for $t < x$, the definition of $\varphi^{(-1)}(y)$ yields $\varphi(t) \leqq y$. This inequality must be strict. Indeed, if $\varphi(t_0) = y$ for some $a \leqq t_0 < x$, then $\varphi(t) = y$

for $t_0 \leqq t < x$, since φ is increasing. The points $(t, \varphi(t))$ are in Φ for $t_0 \leqq t < x$ thus, taking the limit $t \to t_0$, we find that $(x, y) \in \Phi$, i.e., $H(x, y) = f_1(x) + g_1(y)$. This contradicts (21), and proves

$$\varphi(t) < y \quad \text{for} \quad a \leqq t < x. \tag{25}$$

By the properties of Φ there exists a value $a \leqq u \leqq A$ such that $(u, y) \in \Phi$, i.e.,

$$-H(u, y) = -f_1(u) - g_1(y). \tag{26}$$

Then $\varphi(u) \geqq y$, thus (25) implies $x \leqq u$. Applying (17), we have

$$H(u, \varphi(x)) \leqq f_1(u) + g_1(\varphi(x)). \tag{27}$$

Adding the inequalities (21), (22), (26) and (27), we obtain

$$H(x, y) + H(u, \varphi(x)) - H(u, y) - H(x, \varphi(x)) < 0. \tag{28}$$

To get a contradiction we have only to show $\varphi(x) \geqq y$ (since then (28) cannot be valid). If $\varphi(x) = B$ then there is nothing to prove. If $\varphi(x) < B$, then $x < A$. Now $x < t \leqq A$ implies $y \leqq \varphi(t)$. Taking the limit $t \to x+0$ and using the right continuity of φ, we can see that $y \leqq \varphi(x)$. Thus the proof of (20) is complete.

Let $a < t < A$ be an arbitrary point where f_1 is differentiable. Then, by (19), the function

$$x \mapsto f_1(x) + g_1(\varphi(t)) - H(x, \varphi(t))$$

has a minimum at $x = t$. Therefore the derivative vanishes there:

$$f_1'(t) = \partial_1 H(t, \varphi(t)).$$

Since f_1 is absolutely continuous, we have

$$f_1(x) = \int_a^x f_1'(t)\, dt + f_1(a) = \int_a^x \partial_1 H(t, \varphi(t))\, dt + \alpha \tag{29}$$

for all $a \leqq x \leqq A$, where $\alpha = f_1(a)$. Similarly, it follows from (20) that

$$g_1(y) = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds + g_1(b). \tag{30}$$

However, as we have proved, $(a, b) \in \Phi$, that is

$$g_1(b) = H(a, b) - f_1(a) = H(a, b) - \alpha. \tag{31}$$

Since $f = f_1 + p$, $g = g_1 + q$, therefore (29), (30) and (31) show that (12) and (13) are satisfied.

The proof of the theorem is complete.

Remark. In the proof of the "only if" part of Theorem 3 we have not used all the regularity properties of H. We used only inequality (4) and that $\partial_1 H$ and $\partial_2 H$ exist everywhere.

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